Vector quantities

Unit 2

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You have studies vectors in grade 9. An understanding of vectors is essential for an understanding of physics.

They help physicists and engineers to build amazing structures and to design spacecraft, and they also help you find your way home!

2.1 Types of vector

By the end of this section you should be able to:

- Demonstrate an understanding of the difference between scalars and vectors and give common examples.
- Explain what a position vector is.
- Use vector notation and arrow representation of a vector.
- Specify the unit vector in the direction of a given vector.

Introduction and recap of basic vectors

All physical quantities are either scalar or vector quantities:

- A vector quantity has both **magnitude** (size) and direction.
- A **scalar** quantity has magnitude only.

All vector quantities have a direction associated with them. For example, a displacement of 6 km to the West, or an acceleration of 9.81 m/s^2 down. Scalars are just a magnitude; for example, a mass of 70 kg or an energy of 600 J.

Table 2.1 *Some examples of vector and scalar quantities*

Vector quantities	Scalar quantities
Forces (including weight)	Distance
Displacement	Speed
Velocity	Mass
Acceleration	Energy
Momentum	Temperature

KEY WORDS

magnitude *the size of a value*

scalar *a quantity specified only by its magnitude*

Discussion activity

Which of the following do you think are scalars and which are vectors? Electric current, moment, time, potential difference, resistance, volume, air resistance and charge.

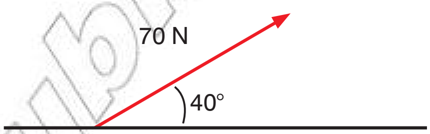

Figure 2.1 *An arrow representing a force of 70 N at about 40° to the horizontal. Is this a vector or a scalar?*

Representing vectors

All vector quantities must include a direction. For example, a displacement of 8 km would not be enough information. We must write 8 km South.

Vectors can be represented by arrows, the magnitude (size) of the vector is shown by the length of the arrow. The direction of the arrow represents the direction of the vector.

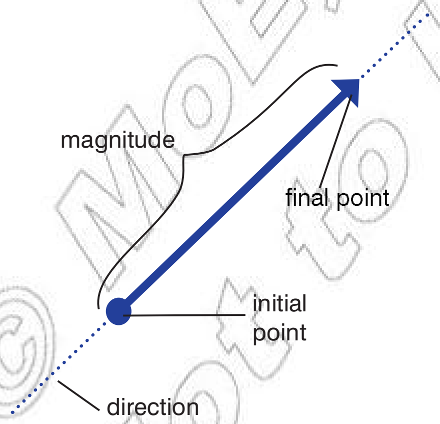

Figure 2.2 *A vector has size (magnitude) and direction.*

DID YOU KNOW?

In 1881 vectors appeared in a publication called *Vector Analysis* by the American J. W. Gibbs. They have been essential to maths and physics ever since!

Representing vectors as arrows

Vectors are sometimes written in lowercase boldface, for example, **a** or ***a***. If the vector represents a displacement from point A to point B, it can also be denoted as:

$\overrightarrow{AB}$

Activity 2.1: Drawing vector diagrams

Draw four vector arrows for the following (you will need to use different scales):

- 140 km North
- 2.2 N left
- 9.81 m/s^2 down
- 87 m/s at an angle of 75° to the horizontal.

Types of vector

There are several different types of vector to consider. These are outlined below.

Position vector

A **position vector** represents the position of an object in relation to another point.

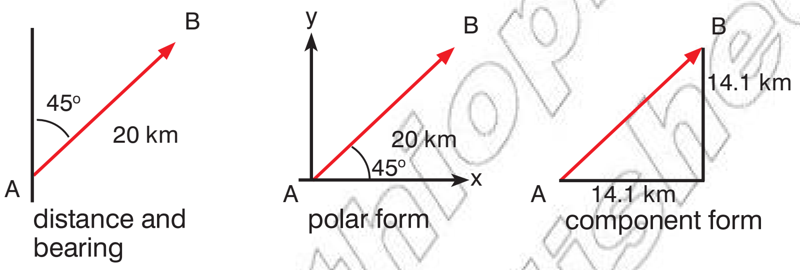

Figure 2.3 *Position vectors*

B is 20 km North East of A. Alternatively this could be written as B is 20 km from A on a bearing of 45°. Remember that bearings are specified as an angle going clockwise from north. The vector can be given in polar form. The angle is given from the positive x-axis, going anticlockwise. The angle can be in degrees or radians. The vector B from A is: B = (20, 45°)

The vector can also be given in component form, where it is given in terms of the components in the x, y and z directions. The vector B is: B = (14.1 km, 14.1 km, 0 km)

Unit vector

A **unit vector** is a vector with a length equal to one unit. For example, Figure 2.4 contains three examples of unit vectors, one each for displacement, force and acceleration.

Unit vectors can also have direction. There are three unit vectors which are used to specify direction, as shown in Figure 2.5:

- unit vector **i** is 1 unit in the x-direction
- unit vector **j** is 1 unit in the y-direction
- unit vector **k** is 1 unit in the z-direction.

Collinear vector

Collinear vectors are vectors limited to only one dimension. Two vectors are said to be collinear if they are parallel to each other and act along the same line. They can be in the same direction or opposite directions.

KEY WORDS

collinear vectors *vectors that are parallel to each other and which act along the same line*

coplanar vectors *vectors that act in the same two-dimensional plane*

position vector *a vector that represents the position of an object in relation to another point*

unit vector *a vector with a length of one unit*

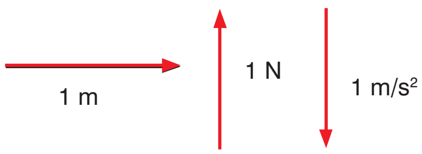

Figure 2.4 *Unit vectors*

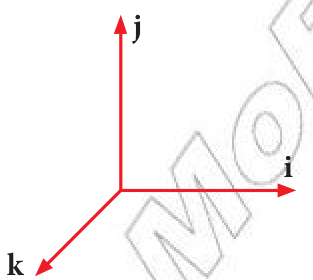

Figure 2.5 *Unit vectors* ***i, j*** *and* **k**

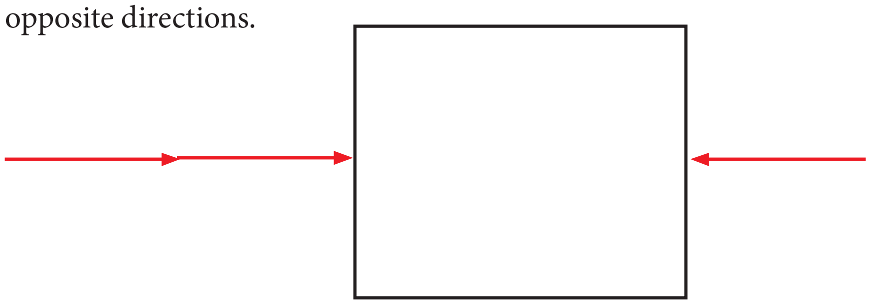

Figure 2.6 *These three vectors are collinear*

Coplanar vector

This refers to vectors in the same two-dimensional plane. This may include vectors at different angles to each other. For example, Figure 2.7 shows two displacement vectors when viewed from above.

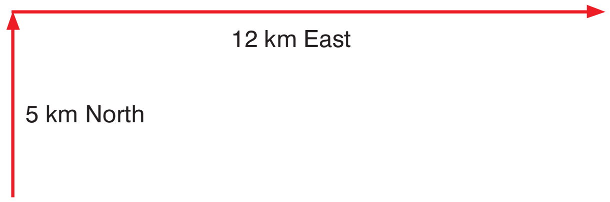

***Figure 2.7** Coplanar displacement vectors*

A more complex example might involve three forces acting on a cube.

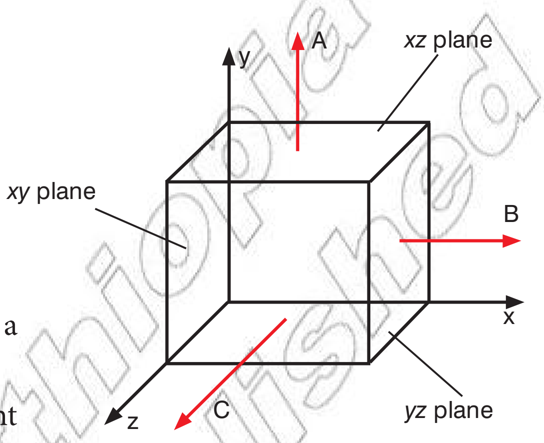

***Figure 2.8** Three vector forces acting on a cube*

A and B are both in the same plane (the *xy* plane) so they might be described as coplanar. C is in a different plane and so is not coplanar.

B and C are in the same plane (the *xz* plane) so they might be described as coplanar. A is in a different plane and so is not coplanar.

A and C are in the same plane (the *yz* plane) so they might be described as coplanar. B is in a different plane and so is not coplanar.

A, B and C cannot be considered to be coplanar with each other as they are in different planes.

Summary

In this section you have learnt that:

- A vector quantity has both magnitude (size) and direction.
- A scalar quantity has magnitude only.
- Vectors are often represented by arrows.
- Different types of vectors include position vectors, unit vectors, collinear vectors (along the same line) and coplanar vectors (in the same two-dimensional plane).

Review questions

1. Define the terms vector and scalar. Give five examples of each.
2. Explain the differences and similarities between position vectors, unit vectors, collinear vectors and coplanar vectors. Give examples for each.

KEY WORDS

component vectors *two or more vectors that, when combined, can be expressed as a single resultant vector*

resolving *splitting one vector into two parts that, when combined, have the same effect as the original vector*

2.2 Resolution of vectors

By the end of this section you should be able to:

- Determine the magnitude and direction of the resolution of two or more vectors using Pythagoras's theorem and trigonometry.

What is resolution?

Resolving means splitting one vector into two **component vectors**. This may be a component in the *x* direction (horizontal) and another in the *y* direction (vertical). The two components have the same effect as the original vector when combined.

An example can be seen in Figure 2.9, the velocity of 25.0 m/s can be resolved into two component vectors that, when combined, have the same effect.

The component vectors can be made to form the sides of a right-angled triangle. They make up the opposite and adjacent sides of the triangle.

As we know the size of the hypotenuse (in this case 25.0 m/s) and the angle (in this case 65°), we can then use trigonometry to find their relative sizes.

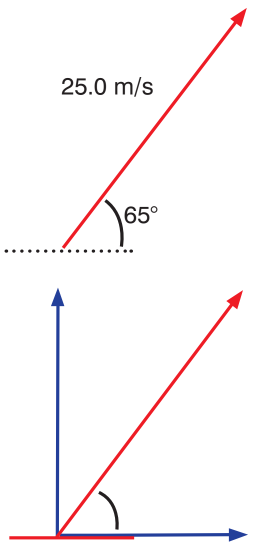

Figure 2.9 *Component vectors of the main vector are shown in blue*

Using trigonometry to resolve vectors

You will probably remember the following rules from your maths class:

$$\sin\theta = \frac{opposite}{hypotenuse}$$

$$\cos\theta = \frac{adjacent}{hypotenuse}$$

$$\tan\theta = \frac{opposite}{adjacent}$$

In the case of Figure 2.9:

- hypotenuse × sin θ = opposite
- 25.0 m/s × sin 65 = 22.7 m/s in the *y* direction
- hypotenuse × cos θ = adjacent
- 25.0 m/s × cos 65 = 10.6 m/s in the *x* direction

You can check your working by using Pythagoras's theorem to recombine the vectors.

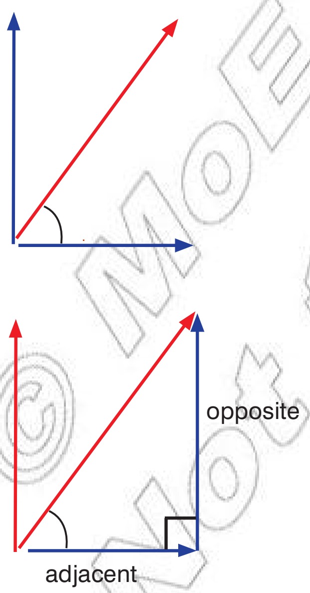

Figure 2.10 *Component vectors as a right-angled triangle*

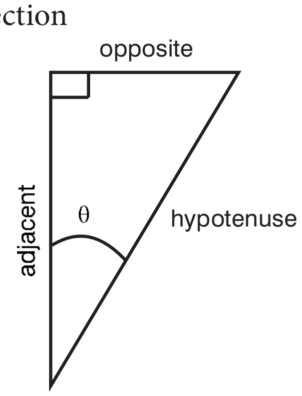

Figure 2.11 *The basic rules of trigonometry*

Pythagoras's theorem

For a right-angled triangle, Pythagoras's theorem states:

$a^2 = b^2 + c^2$

We can use this to work out the magnitude of two coplanar vectors. We can use trigonometry to work out the direction of the two vectors.

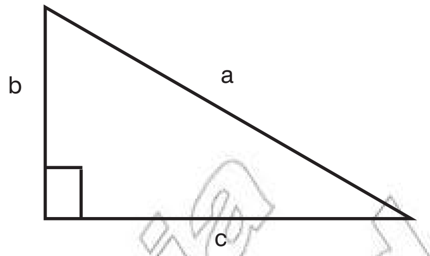

Figure 2.12 *A right-angled triangle demonstrates Pythagoras's theorem*

Worked example 2.1

What is the a) magnitude and b) direction of the two coplanar vectors in Figure 2.13?

a) $D^2 = 3^2 + 4^2 = 9 + 16 = 25$

$D = \sqrt{25} = 5$ m

b) $\tan \theta$ = opposite/adjacent = 4/3 = 1.333 …

$\theta = \tan^{-1} 1.333 \ldots = 53°$

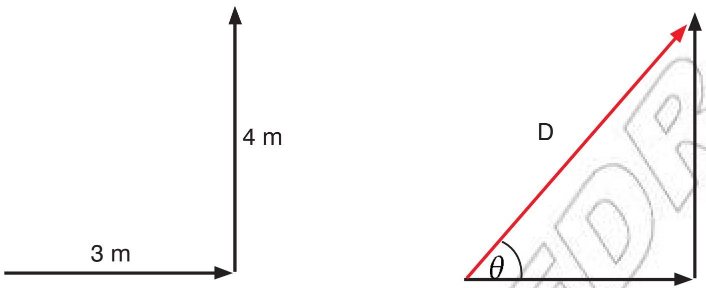

Figure 2.13 *Two perpendicular coplanar vectors form a right-angled triangle.*

Summary

In this section you have learnt that:

- Resolving means splitting a vector into two perpendicular components.
- The components have the same effect as the original vector.
- Trigonometry can be used to determine the magnitude of the components.
- Vectors can be added mathematically using Pythagoras's theorem and trigonometry.

Review questions

1. Explain what it means to resolve a vector.
2. Draw simple vector diagrams and resolve them into two components.
   a) 60 N at an angle of 30° from the horizontal.
   b) 45 m/s at an angle of 80° from the horizontal.
   c) 1900 km at an angle of 40° from the vertical.

DID YOU KNOW?

Vectors are used in computer games to determine the movement of a character. Software will convert the commands from the games controller into a three-dimensional vector to describe how the character should move.

2.3 Vector addition and subtraction

By the end of this section you should be able to:

- Calculate vectors by graphical and mathematical methods.
- Appreciate the parallelogram rule and the triangle rule.
- Solve more complex examples of vectors.

Adding vectors

It is often necessary to add up vectors to find the resultant vector acting on a body. This may be the resultant velocity of an object, the resultant force acting on an object, or even the resultant displacement after several legs of a journey.

Activity 2.2: Drawing scale diagrams

Draw a scale diagram to find the resultant displacement from the following:

- 12 km at an angle of 0° to the vertical
- 24 km at an angle of 90° to the vertical
- 6 km at an angle of 120° to the vertical
- 30 km at an angle of 210° to the vertical

Vector diagrams

The first technique for vector addition involves carefully drawing diagrams. This can only be applied to collinear or coplanar vectors (this is because your diagrams will be two-dimensional only!).

There are three slightly different techniques that could be used.

Scale diagrams

Whenever you are drawing vector diagrams you should draw them to a scale of your own devising. Scale diagrams are very simple:

- Select a scale for your vectors.
- Draw them to scale, one after the other (in any order), lining them up head to tail ensuring the directions are correct.
- The resultant will then be the arrow drawn from the start of the first vector to the tip of the last.

For example:

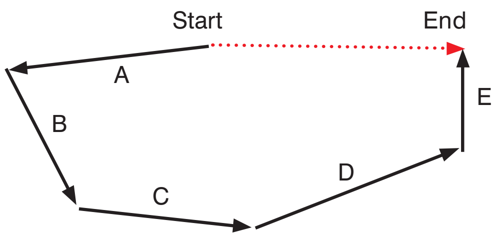

Figure 2.14 *Scale diagram showing a resultant vector (the red arrow) for a series of coplanar vectors*

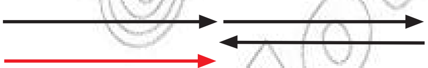

Figure 2.15 *Adding three collinear vectors*

It does not matter in which order you draw your vectors. Check them for yourself!

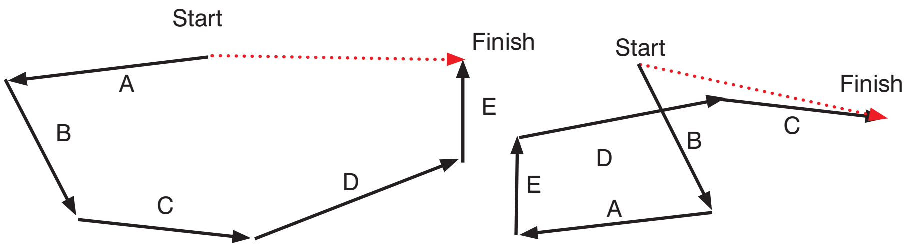

Figure 2.16 *The resultant vector remains the same*

If you end up where you started, then all the vectors cancel out and there is no resultant vector.

The fact that you can add vectors in any order and get the same resultant vector is called the **commutative law**.

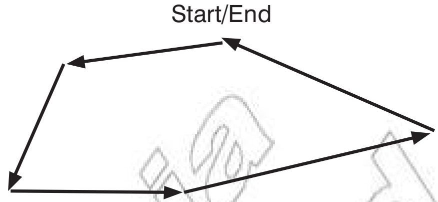

Figure 2.17 *Scale diagram showing no resultant force*

Parallelogram rule

If you have two coplanar vectors, you could use the parallelogram rule. This involves drawing the two vectors with the same starting point. The two vectors must be drawn to a scale and are made to be the sides of the parallelogram. The resultant will be the diagonal of the parallelogram.

Figure 2.18 *Two perpendicular coplanar vectors*

If the vectors are perpendicular, the parallelogram will always be a rectangle.

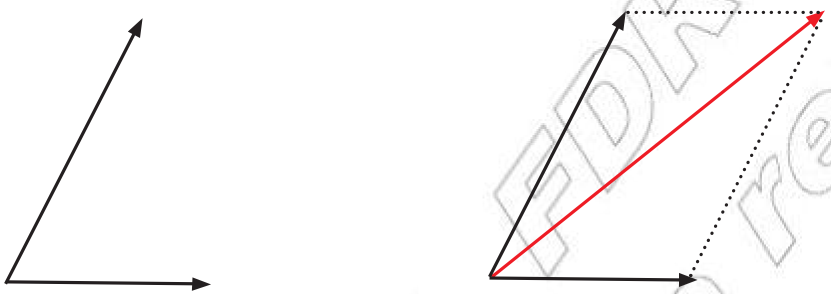

Figure 2.19 *Two non-perpendicular coplanar vectors*

If the vectors are still coplanar but not perpendicular, the parallelogram will not be a rectangle.

Triangle rule

This is a very similar technique, it involves drawing the two coplanar vectors but this time drawing them head to tail. The two vectors must again be drawn to a scale. The resultant will be the missing side from the triangle.

If the vectors are perpendicular, the triangle will be a right-angled triangle.

Figure 2.20 *Two perpendicular coplanar vectors*

KEY WORDS

commutative law *a process obeys the commutative law when it does not matter which order the quantities are in. For example, the addition of numbers obeys the commutative law.*

Activity 2.3: Finding an unknown force

There are three forces acting on an object, A, B and C. This object is at equilibrium (there is no resultant force acting on it). Draw a scale diagram to find the magnitude and direction of the unknown force.

- Force A, 45 N at an angle of 0° to the horizontal
- Force B, 30 N at an angle of 300° to the vertical
- Force C, unknown

Discussion activity

If the triangle is a right-angled triangle, we could use trigonometry to determine the sides and angles mathematically. What if the triangle is not a right-angled triangle?

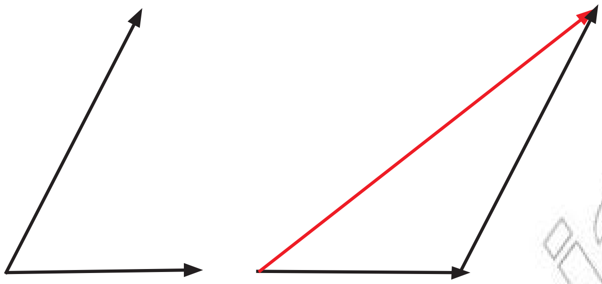

Figure 2.21 *Two non-perpendicular coplanar vectors*

If the vectors are still coplanar but not perpendicular the triangle will not be a right-angled triangle.

Activity 2.4: Determining resultant velocity

Use the parallelogram rule to determine the resultant velocity of the following two velocities:

- Velocity A, 30 m/s at an angle of 45° to the horizontal.
- Velocity B, 40 m/s at an angle of 80° to the horizontal.

Repeat, this time use the triangle rule.

Activity 2.5: Adding vectors

Vector A is 6 m/s along the horizontal and vector B is 9 m/s at 90° to the horizontal.

- Add vector B to vector A.
- Add vector A to vector B.

Does the order you add the vectors make any difference to the resultant?

Adding coplanar vectors mathematically

To add coplanar vectors we use more complex mathematics.

Since two perpendicular coplanar vectors form a right angled-triangle, they can be added using Pythagoras's theorem and trigonometry. Pythagoras's theorem determines the magnitude, and trigonometry can be used to determine the direction.

Component method

We can also express vectors as components in the x, y and z directions.

The resultant vector shown in Figure 2.13 can be expressed in component form as (3, 4), where the first number is the magnitude in the x-direction and the second number is the magnitude in the y-direction. The vector can also be given in the column form $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

We can also add and subtract vectors in this form.

Activity 2.6: Adding forces

You are going to pull on a block of wood with two forces. You will find the resultant of the two forces, and then check your findings by adding the vectors.

- Find a suitable block of wood and three forcemeters (newtonmeters or spring balances). Place the block on a sheet of plain paper.
- Attach two of the forcemeters (A and B) to one end of the block as shown in Figure 2.22. Attach the third (C) to the opposite end.
- One person pulls on each forcemeter. A and B should be at an angle of 90° to each other. C is in the opposite direction. Pull the forcemeters so that their effects balance.
- On the paper, record the magnitudes and directions of the three forces.
- Now add forces A and B together using vector addition.

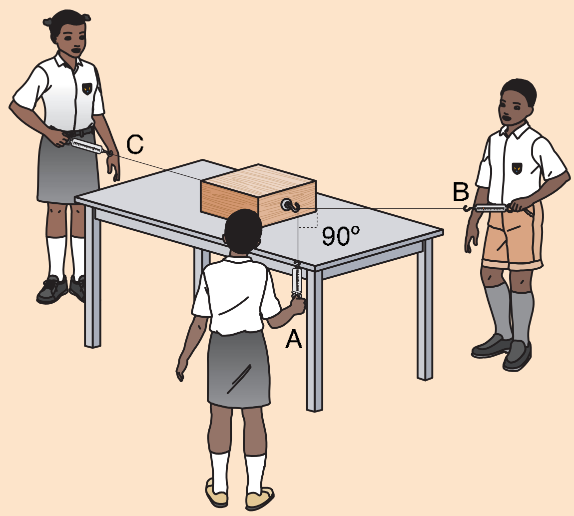

Figure 2.22 *Testing vector addition of forces*

- Because force C balances forces A and B, it must be equal and opposite to the resultant of A and B. Did you find this?
- Repeat the experiment with different forces at a different angle.

Worked example 2.2

Nishan and Melesse are trying to drag a box. Nishan is using a force of (15, 8, 6) N and Melesse is using a force of (12, 10, –6) N.

What is the resultant force on the box?

First draw a diagram to show the forces.

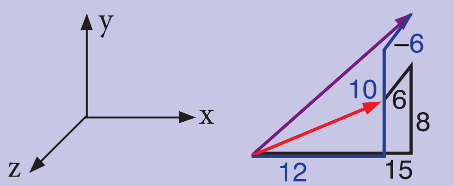

Figure 2.23 *Forces acting on the box*

Add the components of the forces together using:

$F_A + F_B = (A_x + B_x, A_y + B_y, A_x + B_z)$

$F_A = (15, 8, 6)$ N

$F_B = (12, 10, -6)$ N

So $F_A + F_B = (15 + 12, 8 + 10, 6 + -6)$ N

$= (27, 18, 0)$ N

Worked example 2.3

We can also use the cosine rule and the sine rule to work out the magnitude and direction of the sum of two vectors.

Consider two vectors **a** (5 m along horizontal) and **b** (6 m at an angle of 60° above the horizontal).

First draw a diagram showing the vector addition.

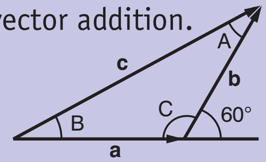

Figure 2.24

We can use the cosine rule to calculate the magnitude of the resultant using

$c = \sqrt{(a^2 + b^2 - 2ab \cos C)}$

where a and b are the magnitudes of the two vectors and c is the magnitudes of the resultant.

We can then use the sine rule to calculate the angle B of the resultant vector c

$\sin B = (b \sin C)/c$

From the diagram, we can see that angle C is 180° – 60° = 120°

Substituting the values in to the equation to find the magnitude of c:

$c = \sqrt{(5^2 + 6^2 - 2 \times 5 \times 6 \cos 120°)}$
$= \sqrt{(25 + 36 - 2 \times 5 \times 6 \times -0.5)}$
$= \sqrt{(25 + 36 + 30)}$
$= \sqrt{91}$
$= 9.54$ m

Substituting into the equation to find the angle

$\sin B = (6 \times \sin 120°)/9.54$
$= 6 \times 0.866/9.54$
$= 0.545$
so $B = \sin^{-1} 0.545 = 33.0°$

So **a** + **b** = 9.54 m at an angle of 33.0° to the horizontal

Summary

In this section you have learnt that:

- Vectors can be added graphically by drawing scale diagrams.
- Vectors can be expressed as components.

Review questions

1. Vector **p** is 6 m in the x-direction. Vector **q** is 10 m in the y-direction.
   a) Use the parallelogram method to work out **p** + **q**.
   b) Use Pythagoras's theorem and trigonometry to work out **p** – **q**.
2. A car travels 3 km due North then 5 km East. Represent these displacements graphically and determine the resultant displacement.
3. Two forces, one of 12 N and another of 24 N, act on a body in such a way that they make an angle of 90° with each other. Find the resultant of the two forces.
4. Two cars A and B are moving along a straight road in the same direction with velocities of 25 km/h and 30 km/h, respectively. Find the velocity of car B relative to car A.
5. Two aircraft P and Q are flying at the same speed, 300 m s^{-1}. The direction along which P is flying is at right angles to the direction along which Q is flying. Find the magnitude of the velocity of the aircraft P relative to aircraft Q.
6. Three vectors are: $\mathbf{a} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 2 \\ -9 \end{bmatrix}$
   Work out the following:
   a) **a** + **b** b) **a** + **c** c) **b** – **c** d) **a** – **c** e) **a** + **b** + **c**
   What does the answer to part e) mean?
7. Work out the magnitude and direction of the resultant force in the worked example on page 31.
8. An aeroplane flies (1500, 3000, 200) m to point A and then (2000, –5000, –100) m to point B.
   a) Work out the final displacement of the aeroplane.
   b) Work out the magnitude of the displacement.
9. Add the following pairs of vectors together, using the cosine and sine rules to work out the resultant vector.
   a) **a** is 4 m due west, **b** is 8 m at an angle of 50° above the horizontal
   b) **a** is 6 m due north, **b** is 4 m at an angle of 30° above the horizontal
   c) **a** is 7 m due west, **b** is 5 m at an angle of 65° below the horizontal

2.4 Multiplication of vectors

By the end of this section you should be able to:

- Use the geometric definition of the scalar product to calculate the scalar product of two given vectors.
- Use the scalar product to determine projection of a vector onto another vector.
- Test two given vectors for orthogonality.
- Use the vector product to test for collinear vectors.
- Explain the use of knowledge of vectors in understanding natural phenomena.

Multiplying by a scalar

Vectors can be multiplied by scalars. When you multiply a vector by a scalar, the magnitude of the vector changes, but not its direction.

Figure 2.25 shows the vector **a**, which has a magnitude of 5 at an angle of 53° to the x-direction. It is multiplied by 2, which gives the vector 2**a**. The diagram shows that the magnitude has doubled but its direction is unchanged.

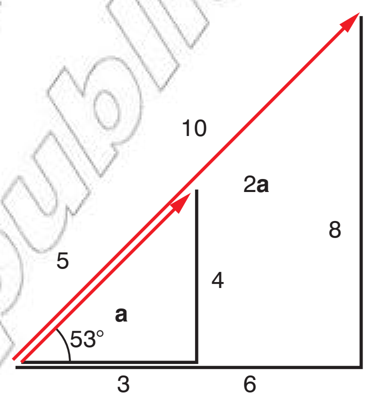

Figure 2.25 *Multiplying a vector by a scalar*

If we break **a** down into its components and express it in column form, it becomes $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Multiplying it by 2 to give 2**a** gives $\begin{bmatrix} 6 \\ 8 \end{bmatrix}$. If θ is the direction of the vector relative to the x-direction, we can see that tan θ is the same for both **a** and 2**a**

For **a** tan θ = 4/3

For 2**a** tan θ = 8/6 = 4/3

If you multiply a vector by a negative scalar, the direction of the vector is reversed.

For example, if **a** is multiplied by –2, then –2**a** is [–6, –8]

Scalar product

The scalar product of two vectors is when they are multiplied together to give a scalar quantity. The scalar product is also known as the dot product.

The scalar product of two vectors is defined as

$$\mathbf{a} \bullet \mathbf{b} = a_x b_x + a_y b_y$$

where the vectors are given in component form and are $\mathbf{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix}$. a_x and b_x are the components in the x-direction and a_y and b_y are the components in the y-direction.

Worked example 2.4

a is the vector $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$ and **b** is the vector $\begin{bmatrix} 7 \\ 3 \end{bmatrix}$.

Work out the scalar product of **a** and **b**.

$\mathbf{a} \bullet \mathbf{b} = a_x b_x + a_y b_y =$

$(4 \times 7) + (4 \times 3) =$

$28 + 12 = 40$

The scalar product can also be expressed as:

$$\mathbf{a} \bullet \mathbf{b} = |a|\,|b| \cos\theta$$

where |a| and |b| are the magnitudes of the vectors **a** and **b**, respectively, and θ is the angle between the two vectors.

By rearranging this equation, we can calculate the angle between two vectors:

$$\cos\theta = \frac{(\mathbf{a} \bullet \mathbf{b})}{(|a|\,|b|)}$$

Worked example 2.5

What is the angle between the two vectors $\mathbf{a} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$?

Draw a sketch of the vectors, as shown in Figure 2.26.

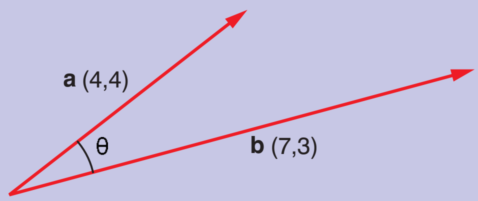

Figure 2.26** Vectors **a** and **b

Work out the magnitudes of **a** and **b** using Pythagoras's theorem.

$|a| = \sqrt{(4^2 + 4^2)} = \sqrt{(16 + 16)} = \sqrt{32}$

$|b| = \sqrt{(7^2 + 3^2)} = \sqrt{(49 + 9)} = \sqrt{58}$

We already know from the previous worked example that a • b = 40

So $\cos\theta = 40/(\sqrt{32} \times \sqrt{58}) = 40/\sqrt{1856} = 0.93$

$\theta = \cos^{-1} 0.93 = 21.8°$

We can also use the scalar product to work out the scalar projection of one vector onto another vector. If the vector **a** is projected on to vector **b** as shown in Figure 2.27, the scalar projection gives the magnitude of the component of **a** that is in the direction of **b**.

The scalar projection of **a** onto **b** is given by:

$$|a| \cos\theta$$

Worked example 2.6

Two vectors are $\mathbf{a} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$. What is the scalar projection of **a** on to **b**?

Sketch the two vectors and the projection of **a** on to **b** as shown in Figure 2.27.

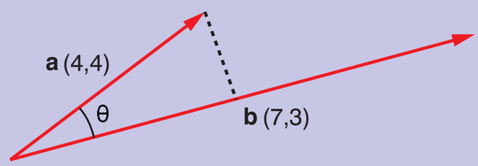

Figure 2.27 *Scalar projection of* **a** *on to* **b**

From worked example 2.5, $|a| = \sqrt{32}$ and $\cos\theta = 0.93$

So the scalar projection of **a** on to **b** is $\sqrt{32} \times 0.93 = 5.25$

Vector product

The vector product of two vectors is when two vectors are multiplied together to produce another vector. If is given by the formula:

$$\mathbf{a} \times \mathbf{b} = |a|\,|b| \sin\theta\, \hat{n}$$

where |a| and |b| are the magnitudes of **a** and **b**, respectively, θ is the smaller angle between **a** and **b** (θ is between 0° and 180°) and n̂ is a unit vector, which is perpendicular to the plane that **a** and **b** are in.

The vector product can also be expressed as:

$$\mathbf{a} \times \mathbf{b} = (a_x b_y - a_y b_x)\,\hat{n}$$

The direction of n̂ is given by the right-hand rule, as shown in Figure 2.28.

If the vectors are in three dimensions, the vector product is a bit more complicated:

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$$

Activity 2.8

The vectors **g** and **h** are $\begin{bmatrix} 7 \\ 0 \end{bmatrix}$ cm and $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ cm, respectively.

a) Draw the vectors to scale.

b) Find the resultant by drawing a parallelogram.

c) Find the area of the parallelogram.

d) Find the vector product of the two vectors

What do you notice?

KEY WORDS

orthogonal *at right angles. When two vectors are at right angles to each other, they are said to be orthogonal.*

Activity 2.7

Consider the vectors

$\mathbf{a} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -9 \\ 6 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 10 \\ -6 \end{pmatrix}$

What is the angle between **a** and **b**?

What is the angle between **c** and **d**?

Can you see an easy way of checking to see if vectors are **orthogonal**?

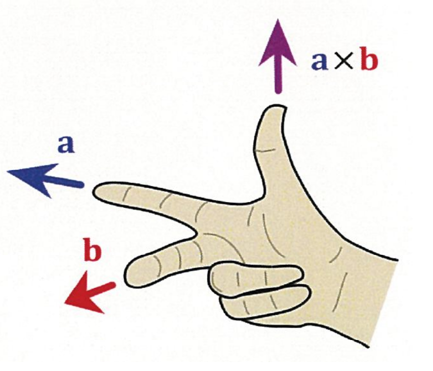

Figure 2.28 *The right-hand rule for finding the direction of the unit vector in the vector product of two vectors*

Project work

A ladder rests against a wall.

Plan and carry out an investigation into the forces being exerted on the ladder.

What directions are they acting in?

Are they in equilibrium?

Write your results up as a report using the writing frame on pages 19–20.

Activity 2.9

1. The vectors **d**, **e** and **f** are $\begin{bmatrix}-4\\1\end{bmatrix}$, $\begin{bmatrix}1\\4\end{bmatrix}$ and $\begin{bmatrix}4\\-1\end{bmatrix}$, respectively.

   Find:

   a) **d** × **e**.

   b) the angle between vectors **d** and **e**.

   c) the area of the parallelogram formed by the resultant of **e** and **f**.

2. When vectors are collinear, they are either in the same direction as each other or in the opposite direction. So, the angle between them will be either 0° or 180°.

   Can you find an easy way to test if vectors are collinear?

Discussion activity

In small groups, discuss other possible applications of vectors.

Report the results of your discussion back to the rest of the class.

We can work out the size of the angle by rearranging the equation for the vector product:

$$\sin\theta = \frac{(\mathbf{a}\times\mathbf{b})}{|a|\,|b|\,\hat{n}}$$

But we also know that **a** × **b** = $(a_x b_y - a_y b_x)\,\hat{n}$, so

$$\sin\theta = \frac{(a_y b_x - a_y b_x)}{|a|\,|b|}$$

Applications of vectors

Vectors have many applications. They are extremely useful in physics and many other areas. Some applications are as follows.

- Analysing forces on a bridge.
- Analysing the motion of an aeroplane.
- Programming motion or the position of an object in a computer game or animation.
- Displaying graphics (in the form of vector graphics) so that the diagram can be resized easily without any loss of quality.
- Modelling and planning the trajectory (path) of a space probe.
- Analysing the motion of planets.
- Analysing magnetic fields.

These are just a few examples – there are many more. We can use vectors whenever there is a variable that has direction as well as magnitude.

Summary

- Multiplying a vector by a scalar changes the magnitude but not the direction of a vector.
- The scalar product of two vectors is **a** • **b** = $a_x b_x + a_y b_y + a_z b_z = |a|\,|b| \cos\theta$
- The vector product of two vectors is **a** × **b** = $|a|\,|b| \sin\theta\,\hat{n} = (a_x b_y - a_y b_x)\,\hat{n}$.

Review questions

1. Vector **d** is 1 N at 50° to the x-direction.

   Vector **e** is 3 N in the x-direction and 2 N in the y-direction.

   Vector **f** is (6, 2).

   Work out the following:

   a) **2d**
   b) 3**e**
   c) −2f
   d) 1/2**d**
   e) −3/4**e**
   f) 0.25**f**

2. Four vectors are $\mathbf{a} = \begin{bmatrix}4\\10\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}2\\-5\end{bmatrix}$, $\mathbf{c}\begin{bmatrix}-6\\15\end{bmatrix}$, $\mathbf{d}\begin{bmatrix}15\\-6\end{bmatrix}$
   a) Work out:
      i) $\mathbf{a} \bullet \mathbf{b}$
      ii) $\mathbf{c} \bullet \mathbf{d}$
      iii) the angle between **c** and **d**
      iv) the projection of **c** on to **a**
   b) Are the vectors **a** and **d** orthogonal?
3. a) Express the unit vectors **i**, **j** and **k** in column form.
   b) Work out $\mathbf{i} \bullet \mathbf{j}$, $\mathbf{j} \bullet \mathbf{k}$ and $\mathbf{i} \bullet \mathbf{k}$. What does this tell you about the unit vectors?
4. Work out $\mathbf{i} \times \mathbf{i}$, $\mathbf{j} \times \mathbf{j}$ and $\mathbf{k} \times \mathbf{k}$. Explain your answers.

End of unit questions

1. Construct a glossary of the key terms in this unit. You could add it to the one you made for Unit 1.
2. What is the scalar product of two vectors?
3. Given that $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, find
   a) $\mathbf{a} \bullet \mathbf{b}$
   b) the included angle between the vectors to 1d.p
4. If $\mathbf{p} = 4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{q} = 2\mathbf{i} - \mathbf{j} + 11\mathbf{k}$, find
   a) $\mathbf{p} \bullet \mathbf{q}$
   b) $\mathbf{q} \bullet \mathbf{p}$
5. If $\mathbf{x} = \begin{bmatrix}2\\-3\\-4\end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix}-2\\14\\1\end{bmatrix}$, calculate
   a) $\mathbf{x} \bullet \mathbf{y}$
   b) $\mathbf{y} \bullet \mathbf{x}$
6. What is the vector product of two vectors?
7. Find the vector product $\mathbf{a} \times \mathbf{b}$ if $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}$.
8. If $\mathbf{a} = 8\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ show that
   a) $\mathbf{a} \times \mathbf{b} = -5\mathbf{i} - 18\mathbf{j} - 29\mathbf{k}$
   b) $\mathbf{b} \times \mathbf{a} = 5\mathbf{i} + 18\mathbf{j} + 29\mathbf{k}$
9. How can you test to see if vectors are:
   a) orthogonal b) collinear?
10. Give some applications of vectors.